2.7
Rational Zeros Theorem, Finding Rational Zeros, Upper & Lower Bounds, Descartes' Rule of Signs

Rational Zeros Theorem: Real zeros of polynomial functions are either rational zeros (rational numbers) or irrational zeros (irrational numbers).

Suppose f is a polynomial function of degree $n \geq 1$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0,$$

with every coefficient an integer and $a_0 \neq 0$. If $x = p/q$ is a rational zero of f, where p and q have no common integer factors other than 1, then

- p is an integer factor of the constant coefficient a_0, and
- q is an integer factor of the leading coefficient a_n.

Example:

Find all the possible rational zeros of $f(x) = 3x^3 + 4x^2 - 5x - 2$.

Potential rational zeros: $\dfrac{factors\ \ of\ \ -2}{factors\ \ of\ \ 3} = \dfrac{\pm 1, \pm 2}{\pm 1, \pm 3} = \pm 1, \pm 2, \pm \dfrac{1}{3}, \pm \dfrac{2}{3}$

Now how do we find **all** rational zeros of the polynomial?

Upper and Lower bound tests for Real Zeros:

Let f be a polynomial function of degree $n \geq 1$ with a positive leading coefficient. Suppose $f(x)$ is divided by $x - k$ using synthetic division.

- If $k \geq 0$ and every number in the last line is nonnegative (positive or zero), then k is an **upper bound** for the real zeros of f.

- If $k \le 0$ and the numbers in the last line are alternately nonnegative and nonpositive, then k is a **lower bound** for the real zeros of *f*.

Example:

Prove that all of the real zeros *of* $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$ lie in the interval [-2,5].

(Prove that -2 is a lower bound and 5 is an upper bound using synthetic division.)

The French mathematician Renè Descartes (1596-1650) recognized a connection between the roots of a polynomial equation and the + and – signs of the standard form.

Descartes' Rule of Signs:

Let P(*x*) be a polynomial with real coefficients written in standard form.

- The number of positive real roots of P(*x*) = 0 is either equal to the number of sign changes between consecutive coefficients of P(x) or is less than that by an even number.
- The number of negative real roots of P(*x*) = 0 is either equal to the number of sign changes between consecutive coefficients of P(-*x*) or is less than that by an even number.

  In both cases, count multiple roots according to their multiplicity.

Example using Descartes' Rule of Signs:

What does Descartes's Rule of Signs tell you about the real roots of

$2x^4 - x^3 + 3x^2 - 1 = 0$?

Answer:

$P(x) = 2x^4 - x^3 + 3x^2 - 1 = 0$ *has* 3 *sign changes between terms, therefore there are either* 3 *or* 1 *positive real roots.*

$P(-x) = 2x^4 + x^3 + 3x^2 - 1 = 0$ *has* 1 *sign change between terms, therefore there is* 1 *negative real root.*